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NONCLASSICAL HEAT CONDUCTION PROBLEM FOR CRYSTAL BODIES WITH COATINGS HEATED BY RADIATION WITH ACCOUNT FOR THE CURVATURE OF THE COATING

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Generalized conditions for heat transfer of thermosensitive crystal bodies with coatings are formulated that are necessary for determination of their Kirchhoff variables with account for the curvature of the coating.

Since the first step in the determination of thermal stresses in high-rate unsteady-state processes is preliminary determination of the temperature field, it seems of interest to consider some problems of generalized heat conduction of a thermosensitive inhomogeneous body.

Generalized boundary conditions for heat transfer of bodies with thin coatings and generalized conditions of thermal contact between heterogeneous bodies are formulated in [1–3], neglecting the curvature of the interlayer surface.

In the present work we will consider crystal bodies with coatings whose thermal properties are proportional to the cube of the absolute temperature [4]. In terms of the Kirchhoff variable the heat conduction problem for such an inhomogeneous body can be fully linearized even under complicated nonlinear boundary conditions for radiation. Then, using the operator method and the limiting transition technique, it is possible to exclude the thin coating from consideration; its effect is characterized by complicated nonclassical boundary conditions that contain reduced thermophysical parameters of the coating [4].

Let a crystal body be coated by a thin crystal body of a different material, whose thermal conductivities and volumetric specific heats are different and are proportional to the cube of the absolute temperature at temperatures below the Debye temperature [4, 5]:

$$\lambda_i(t_i) = \tilde{\kappa}_i t_i^3, \quad c_i(t_i) = \tilde{\beta}_i t_i^3, \quad i = 0.1\,. \tag{1}$$

The system is heated by internal heat sources and radiation and the conditions of ideal thermal contact hold between the body and the coating. In this case for determination of the generalized temperature field in the investigated piecewise-homogeneous body in the system of curvilinear orthogonal coordinates (α, β, γ) we have the generalized heat conduction equation [6–8]

$$\frac{1}{H_1H_2H_3}\left\{\frac{\partial}{\partial\alpha}\left[\frac{H_2H_3}{H_1}\lambda_i(t_i)\frac{\partial t_i}{\partial\alpha}\right]+\frac{\partial}{\partial\beta}\left[\frac{H_1H_3}{H_2}\lambda_i(t_i)\frac{\partial t_i}{\partial\beta}\right]+\right.$$
$$\left.+\frac{\partial}{\partial\gamma}\left[\frac{H_1H_2}{H_3}\lambda_i(t_i)\frac{\partial t_i}{\partial\gamma}\right]\right\} = l_i\left[c_i(t_i)\,t_1 - \omega_i\right], \tag{2}$$

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the generalized conditions of ideal contact

$$t_0 = t_1 , \quad \frac{1}{\tau_r^{(0)}} \int_0^\tau \tilde{\varphi}_0 (\xi , \tau)\, d\xi = \frac{1}{\tau_r^{(1)}} \int_0^\tau \tilde{\varphi}_1 (\xi , \tau)\, d\xi \quad \text{at} \quad \gamma = - \delta , \tag{3}$$

the generalized Stefan-Boltzmann boundary conditions

$$\lambda_0 (t_0) \frac{\partial t_0}{\partial \gamma} + l_0 (\tilde{\sigma}_0 t_0^4 - q_0) = 0 \quad \text{at} \quad \gamma = + \delta \tag{4}$$

and the initial conditions

$$t_i = t_0^{(0)} , \quad \dot{t}_i = 0 \quad \text{at} \quad \tau = 0 , \tag{5}$$

where

$$l_i = 1 + \tau_r^{(i)} \frac{\partial}{\partial \tau} , \quad \tilde{\varphi}_i (\xi , \tau) = \lambda_i (t_i) \exp \left(\frac{\xi - \tau}{\tau_r^{(i)}} \right) \frac{\partial t_i}{\partial \gamma} , \quad \dot{t}_i = \frac{\partial t_i}{\partial \tau} .$$

With relations (1), boundary-value problem (2) – (5) for crystalline bodies is linearized completely in terms of the Kirchhoff variables

$$\vartheta_i = \frac{1}{\kappa_i} \int_0^{t_i} \lambda_i (\xi)\, d\xi$$

For the coating in curvilinear mixed coordinates, after suitable simplifications [2] the problem becomes

$$\frac{\partial^2 \vartheta_0}{\partial \gamma^2} + 2k \frac{\partial \vartheta_0}{\partial \gamma} + p^2 \vartheta_0 = - l_0 \frac{w_0}{\kappa_0} , \tag{6}$$

$$\vartheta_0 = \vartheta_1 , \quad \frac{\kappa_0}{\tau_r^{(0)}} \int_0^\tau \varphi_0 (\xi , \tau)\, d\xi = \frac{\kappa_1}{\tau_r^{(1)}} \int_0^\tau \varphi_1 (\xi , \tau)\, d\xi \quad \text{at} \quad \gamma = - \delta , \tag{7}$$

$$\kappa_0 \frac{\partial \vartheta_0}{\partial \gamma} + l_0 (\sigma_0 \vartheta_0 - q_0) = 0 \quad \text{at} \quad \gamma = + \delta , \tag{8}$$

$$\vartheta_0 = \vartheta_0^{(0)} , \quad \dot{\vartheta}_0 = 0 \quad \text{at} \quad \tau = 0 , \tag{9}$$

where

$$\dot{\vartheta}_0 = \frac{\partial \vartheta_0}{\partial \tau} ; \quad p^2 = \Delta - \frac{\beta_0}{\kappa_0} l_0 \frac{\partial}{\partial \tau} , \quad \sigma_0 = 4 \tilde{\sigma}_0 \frac{\kappa_0}{\tilde{\kappa}_0} ;$$

$$\Delta = \frac{1}{AB} \left[\frac{\partial}{\partial \alpha} \left(\frac{B}{A} \frac{\partial}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{A}{B} \frac{\partial}{\partial \beta} \right) \right] ;$$

$$\varphi_i (\xi , \tau) = \exp \left(\frac{\xi - \tau}{\tau_r^{(i)}} \right) \frac{\partial \vartheta_i}{\partial \gamma} .$$

Using the operator method, a general solution of Eq. (6) with the first condition of (7) can be found as

$$\vartheta_0 = e^{-k\gamma}\left[\frac{E\cos\bar{p}\gamma + F\sin\bar{p}\gamma}{\left(-k+\dfrac{\sigma_0}{\kappa_0}\right)\sin 2\bar{p}\delta + \bar{p}\cos 2\bar{p}\delta} + \frac{l_0 Q_0}{\kappa_0 \bar{p}^2}\right], \tag{10}$$

where

$$E = (\bar{p}\cos\bar{p}\delta - k\sin\bar{p}\delta)\, t_1 e^{-k\delta} + \frac{\sigma_0 l_0}{\kappa_0}(e^{k\delta}\, t_c + e^{-k\delta}\, t_1)\sin\bar{p}\delta - \frac{l_0 q_{01}}{\kappa_0\bar{p}^2};$$

$$F = (k\cos\bar{p}\delta + \bar{p}\sin\bar{p}\delta)\, t_1 e^{-k\delta} + \frac{\sigma_0 l_0}{\kappa_0}(e^{k\delta}\, t_c - e^{-k\delta}\, t_1)\cos\bar{p}\delta - \frac{l_0 q_{02}}{\kappa_0\bar{p}^2};$$

$$q_{01} = \bar{p}\cos\bar{p}\delta Q_0^- + \left(\frac{\partial Q_0}{\partial\gamma}\right)^+ \sin\bar{p}\delta + \frac{\sigma_0 l_0}{\kappa_0}(Q_0^+ + Q_0^-)\sin\bar{p}\delta;$$

$$q_{02} = \bar{p}\sin\bar{p}\delta Q_0^- + \left(\frac{\partial Q_0}{\partial\gamma}\right)^- \cos\bar{p}\delta + \frac{\sigma_0 l_0}{\kappa_0}(Q_0^+ - Q_0^-)\cos\bar{p}\delta;$$

$$Q_0 = \bar{p}\int_0^{\gamma}\sin\bar{p}(\xi-\gamma)\, w_0(\xi,\tau)\, d\xi; \quad Q_0^{\pm} = Q_0|_{\gamma=\pm\delta};$$

$$\left(\frac{\partial Q_0}{\partial\gamma}\right)^{\pm} = \left(\frac{\partial Q_0}{\partial\gamma}\right)_{\gamma=\pm\delta}; \quad \bar{p}^2 = p^2 - k^2.$$

Substituting solution (10) into the integral characteristics of the Kirchhoff variable for the coating [9]

$$\theta_0 = \frac{1}{2\delta}\int_{-\delta}^{\delta}\vartheta_0 d\gamma, \quad \theta_0^* = \frac{3}{2\delta^2}\int_{-\delta}^{\delta}\gamma\vartheta_0 d\gamma, \tag{11}$$

we obtain

$$\theta_0 = \frac{E(k\cos\bar{p}\delta\,\mathrm{sh}\,k\delta + \bar{p}\sin\bar{p}\delta\,\mathrm{ch}\,k\delta) - F(k\sin\bar{p}\delta\,\mathrm{ch}\,k\delta - \bar{p}\cos\bar{p}\delta\,\mathrm{sh}\,k\delta)}{\delta(k^2+\bar{p}^2)\left[\left(-k+\dfrac{\sigma_0 l_0}{\kappa_0}\right)\sin 2\bar{p}\delta + \bar{p}\cos 2\bar{p}\delta\right]} + \frac{l_0\tilde{Q}_0}{\kappa_0\bar{p}^2}, \tag{12}$$

where

$$\theta_0^* = 3\frac{EE^* - FF^*}{\delta^2(k^2+\bar{p}^2)\left[\left(-k+\dfrac{\sigma_0 l_0}{\kappa_0}\right)\sin 2\bar{p}\delta + \bar{p}\cos 2\bar{p}\delta\right]} + \frac{l_0\tilde{Q}_0^*}{\kappa_0\bar{p}^2},$$

$$E^* = -k\delta\cos\bar{p}\delta\,\mathrm{ch}\,k\delta + m\cos\bar{p}\delta\,\mathrm{sh}\,k\delta - \bar{p}\delta\,\mathrm{sh}\,k\delta\sin\bar{p}\delta + n\sin\bar{p}\delta\,\mathrm{ch}\,k\delta;$$

$$F^* = -k\delta\sin\bar{p}\delta\,\mathrm{sh}\,k\delta + m\sin\bar{p}\delta\,\mathrm{ch}\,k\delta + \bar{p}\delta\cos\bar{p}\delta\,\mathrm{ch}\,k\delta - n\cos\bar{p}\delta\,\mathrm{sh}\,k\delta;$$

$$m = \frac{k^2-\bar{p}^2}{k^2+\bar{p}^2}; \quad n = \frac{2k\bar{p}}{k^2+\bar{p}^2}; \quad \tilde{Q}_0 = \frac{1}{2\delta}\int_{-\delta}^{\delta}Q_0 d\gamma; \quad \tilde{Q}_0^* = \frac{3}{2\delta^2}\int_{-\delta}^{\delta}\gamma Q_0 d\gamma.$$

Equation (6) will be averaged in accordance with (11) and with account for the second condition in (7) and expressions (12). In the resultant relation the limit of vanishing δ will be taken, with the reduced quantities $\Lambda_0 = 2\delta\kappa_0$, $C_0 = 2\delta\beta_0$, $r_0 = 2\delta/\kappa_0$, W_0, and W_0^* being held constant. As a result, we arrive at the following condition of heat transfer through a thin coating:

$$\Lambda_0\Lambda\left\{\left(1+\frac{\sigma_0 l_0+2k\kappa_0}{2h}\right)t_1+\frac{\kappa_1}{2h\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial t_1}{\partial\gamma}+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)K(\tau)\right]\right\}+$$

$$+\sigma_0 l_0\left(1+\frac{k\kappa_0}{h}\right)(t_c-t_1)-\left(1+\frac{l_0\sigma_0}{h}\right)\times\frac{\kappa_1}{\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial t_1}{\partial\gamma}+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)K(\tau)\right]=$$

$$=C_0 l_0\left\{\left(1+\frac{l_0\sigma_0+2k\kappa_0}{2h}\right)\times\frac{\partial t_1}{\partial\tau}+\frac{\kappa_1}{2h\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial^2 t_1}{\partial\gamma\partial\tau}+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)\times\right.\right.$$

$$\left.\left.\times\left(\frac{\partial t_1}{\partial\gamma}-\frac{1}{\tau_r^{(1)}}K(\tau)\right)\right]\right\}+l_0\overline{W}_0, \tag{13}$$

where

$$\overline{W}_0=W_0-\frac{1}{3}W_0^*;\quad r_0=1/h,\quad K(\tau)=\int_0^\tau\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi;$$

$$W_0=\int_{-\delta}^{\delta}w_0 d\gamma;\quad W_0^*=\frac{3}{\delta}\int_{-\delta}^{\delta}\gamma w_0 d\gamma.$$

We proceed to particular cases of condition (13).

1. Let $\tau_r^{(0)}\to 0$, $\tau_r^{(1)}\neq 0$; then we have the condition

$$\Lambda_0\Lambda\left[\left(1+\frac{\sigma_0+2k\kappa_0}{2h}\right)t_1+\frac{\kappa_1}{2h\tau_r^{(1)}}K(\tau)\right]+\sigma_0\left(1+\frac{k\kappa_0}{h}\right)(t_c-t_1)-$$

$$-\left(1+\frac{\sigma_0}{h}\right)\frac{\kappa_1}{\tau_r^{(1)}}K(\tau)=C_0\left[\left(1+\frac{\sigma_0+2k\kappa_0}{2h}\right)\times\right.$$

$$\left.\times\frac{\partial t_1}{\partial\tau}+\frac{\kappa_1}{2h\tau_r^{(1)}}\left(\frac{\partial t_1}{\partial\gamma}-\frac{1}{\tau_r^{(1)}}K(\tau)\right)\right]+\overline{W}_0. \tag{14}$$

2. If $\tau_r^{(0)}\neq 0$, $\tau_r^{(1)}\to 0$, then

$$\Lambda_0\Lambda\left[\left(1+\frac{\sigma_0 l_0+2k\kappa_0}{2h}\right)t_1+\frac{\kappa_1 l_0}{2h}\frac{\partial t_1}{\partial\gamma}\right]+\sigma_0 l_0\left(1+\frac{k\kappa_0}{h}\right)\times$$

$$\times(t_c-t_1)-\left(1+\frac{\sigma_0 l_0}{h}\right)\kappa_1 l_0\frac{\partial t_1}{\partial\gamma}=C_0 l_0\left[\left(1+\frac{\sigma_0+2k\kappa_0}{2h}\right)\times\frac{\partial t_1}{\partial\gamma}+\frac{\kappa_1 l_1}{2h}\frac{\partial^2 t_1}{\partial\gamma\partial\tau}\right]+l_0\overline{W}_0. \tag{15}$$

3. For the classical case ($\tau_r^{(0)}\to 0$, $\tau_r^{(1)}\to 0$), the following relation is obtained:

$$\Lambda_0\Lambda\left[\left(1+\frac{\sigma_0+2k\kappa_0}{2h}\right)t_1+\frac{\kappa_1}{2h}\frac{\partial t_1}{\partial y}\right]+\sigma_0\left(1+\frac{k\kappa_0}{h}\right)(t_c-t_1)-$$

$$-\kappa_1\left(1+\frac{\sigma_0}{h}\right)\frac{\partial t_1}{\partial y}=C_0\left[\left(1+\frac{\sigma_0+2k\kappa_0}{2h}\right)\frac{\partial t_1}{\partial \tau}+\frac{\kappa_1}{2h}\frac{\partial^2 t_1}{\partial y\partial\tau}\right]+\overline{W}_0. \qquad (16)$$

4. For metals ($\tau_r^{(0)}=\tau_r^{(1)}$) we have

$$\Lambda_0\Lambda\left[\left(1+\frac{\sigma_0 l_0+2k\kappa_0}{2h}\right)t_1+\frac{\kappa_1}{2h}\frac{\partial t_1}{\partial y}\right]+\sigma_0 l_0\left(1+\frac{k\kappa_0}{h}\right)(t_c-t_1)-$$

$$-\kappa_1\left(1+\frac{\sigma_0 l_0}{h}\right)\frac{\partial t_1}{\partial y}=C_0 l_0\left[\left(1+\frac{\sigma_0 l_0+2k\kappa_0}{2h}\right)\frac{\partial t_1}{\partial \tau}+\frac{\kappa_1}{2h}\frac{\partial^2 t_1}{\partial y\partial\tau}\right]+l_0\overline{W}_0. \qquad (17)$$

If the products of $\Lambda_0 r_0$ and $C_0 r_0$ are neglected in conditions (13)–(17), we obtain simpler conditions. In particular, conditions (13) and (16) will be written as

$$\Lambda_0\Delta t_1+\sigma_0 l_0\left(1+\frac{k\kappa_0}{h}\right)(t_c-t_1)-\left(1+\frac{\sigma_0 l_0}{h}\right)\frac{\kappa_1}{\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial t_1}{\partial y}+\right.$$

$$\left.+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)K(\tau)\right]=C_0 l_0\frac{\partial t_1}{\partial\tau}+l_0\overline{W}_0, \qquad (18)$$

$$\Lambda_0\Delta t_1+\sigma_0\left(1+\frac{k\kappa_0}{h}\right)(t_c-t_1)-\kappa_1\left(1+\frac{\sigma_0}{h}\right)\frac{\partial t_1}{\partial y}=C_0\frac{\partial t_1}{\partial y}+\overline{W}_0. \qquad (19)$$

The complicated boundary conditions (13)–16) of heat transfer through a thin coating are characterized by the reduced thermophysical properties of the coating. The region occupied by the coating could be excluded from consideration; its effect is characterized by complicated boundary conditions containing derivatives in the time and space coordinates of the first and second orders. These boundary conditions differ from the ordinary boundary conditions used in classical mathematical physics and are nonclassical. Consequently, concrete heat conduction problems solved using these conditions are called nonclassical.

NOTATION

t_i, w_i, $\tau_r^{(i)}$ ($i=0, 1$), temperature, density of heat sources, and time of relaxation of the heat flux of the coating and the body; $\lambda(t_i)$, κ_i, $c_i(t_i)$, their thermal conductivities, reference thermal conductivities, and volumetric specific heats, respectively; $\tilde{\sigma}$, apparent radiative heat transfer coefficient; q_0, heat flux emitter of the coating; H_1,H_2, H_3, Lamé coefficients; A, B, coefficients of the first quadratic form of the middle surface of the coating; τ, time; W_0, W_0^*, density of heat sources and density of "moments" of heat sources, characterizing the nonuniformity of the distribution of the sources over the interlayer thickness, per unit area of the middle plane of the layer; k, curvature of the middle surface of the coating.

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